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The total area swept out by AB is ls or $l^2\phi$. This gives the exact equation,

$$[5] \qquad l^2\phi = C - ADE + EFH - HIK.$$

If the starting point for tracing the figure be taken as above (*i. e.* at the center of gravity of the area), the algebraic sum of the three areas enclosed by the curve of pursuit and the arc KA will be very nearly zero and we have the approximate equation

$$[6] \qquad c = l^2\phi = l.\mathrm{arc}\,KA.$$

If the angle ϕ be small, say less than 20°, the arc KA can be replaced by its chord, and we have

$$[7] \qquad c = l.KA.$$

If an area whose longest diameter is four inches or less, be measured with a ten inch planimeter, the error is very small and is about equal to the error made in finding the area of an equivalent rectangle by measuring the sides with a scale. The error due to the non-alignment of the edge and tracing point can be eliminated by tracing the curve in opposite directions and finding the mean of the two results.

An improved planimeter of this type has a small chisel-edged wheel instead of the chisel edge.

EXISTENCE OF A MINIMUM OF A QUADRATIC FUNCTION.

By T. H. HILDEBRANDT, The University of Chicago.

Suppose we have a quadratic function in n variables,

$$F[x_1, \ldots, x_n] = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i\, x_j + \sum_{i=1}^{n} b_i\, x_i + c,$$

of which we know that

$$F[x_1, \ldots, x_n] \geqq 0,$$

for all values of the variables $x_1, \ldots, x_n$. For instance, such a function is

$$\int_0^{2\pi} [\phi(x) - \sum_{i=1}^{n} (a_i \cos ix + b_i \sin ix)]^2 dx,$$

where the a_i and b_i take the place of x_i. It might appear that because the function has a lower bound, it also has a minimum, *i. e.*, reaches its greatest lower bound. But this statement is not sufficient to establish the fact that it reaches its greatest lower bound in a finite point. If the x_i were restricted to be finite, we could apply the theorem on continuous functions of n variables, which states that in such a case the function has a minimum. But the theorem does not apply when the x_i are permitted to assume infinite values also. We proceed to show that even in this case there is at least one point in the finite part of n-space at which F takes on its minimum value.

Transform F by a linear homogeneous substitution:

$$x_i = \sum_{j=1}^{n} a_{ij}\, x'_i\, x'_j \qquad i=1, \ldots, n,$$

of such a nature that the determinant $| a_{ij} |$ of the substitution is different from zero and so that in the transformed function f the cross product terms $x'_i\, x'_j$ disappear. This is always possible, as a matter of fact, even in the special case where the substitution is orthogonal and therefore $| a_{ij} | = 1$. Our function then takes the form:

$$\sum_{i=1}^{n} a'_i\, x'^2_i + \sum_{i=1}^{n} b'_i\, x'_i + c = f(x'_1, \ldots, x'_n),$$

while $f(x'_1, \ldots, x'_n) = F(x_1, \ldots, x_n) \geqq 0$. Then we have $a'_i \geqq 0 \quad (i=1, \ldots, n)$. For, suppose $a_i' < 0$. Then let $x'_1, \ldots, x'_{j-1}\, x'_{j+1}, \ldots, x'_n$ take the zero values. We have

$$f(0, \ldots, 0, x_j, 0, \ldots, 0) = a_j'\, x_j^2 + b_j\, x_j + c.$$

But evidently since $a'_j < 0$ we may choose x'_j so large that this becomes negative, contrary to the hypothesis that f shall be greater than or equal to zero for all values of the x'_j. Moreover, if $a'_j = 0$, then $b'_j = 0$. Suppose $a'_j = 0$ and $b'_j < 0$. Then by taking for $x'_1. ,\ldots, x'_{j-1}, x'_{j+1}, \ldots, x'_n$, the values zero, we can find a sufficiently large positive value for x'_j to make $b'_j\, x'_j + c'$ negative, and hence, f negative. Hence, if $a'_j = 0$, then $b'_j = 0$. Our function is then of the form:

$$f(x'_1, \ldots, x_r') \equiv \sum_{i-1}^{r} [a'_i\, x'^2_i + b'_i\, x'_i + c] \qquad (r \leqq n).$$

Transform this function by the substitution

$$x_i'' = x'_i + \frac{b_i}{2a_i}.$$

Then we obtain

$$F(x_1, \ldots, x_n) \equiv \phi(x_1'', \ldots, x_n'') \equiv \sum_{i=1}^{r} a_i' \, x_i''^2 + c' \geqq 0.$$

Evidently $c' \geqq 0$. Moreover we now see at once the minimum value of ϕ, namely,

$$x_1''=0, \ldots, x_n''=0.$$

By putting $x_i''=0$ and solving our transformations backward we can obtain the values of $x_1, \ldots, x_n$ which will make $F(x_1, \ldots, x_n)$ a minimum. There will be a unique solution if $r=n$. For, then we will have n equations in n unknowns. If $r<n$, the solution is not unique, but there will be at least a single infinity of values which will make F a minimum. When $r=n$, our function f evidently has the form:

$$f(x'_1, \ldots, x'_n) = \sum_{i=1}^{n} (a'_i \, x'^2_i + b'_i \, x'_i + c).$$

Now $\sum_{i=1}^{n} a'_i \, x'^2_i$ is a positive, definite form, and so $\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \, x_i \, x_j$ must also be. That is, the condition that

$$F(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \, x_i \, x_j + \sum_{i=1}^{n} b_i \, x_i + c \geqq 0$$

may have a minimum is that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \, x_i \, x_j$$

be a positive definite form. This is exactly the result obtained by applying the methods of the differential calculus.